Boundary behavior of the iterates of a self-map of the unit disk

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Let $M$ be a complex manifold and $\varphi : M \to M$ holomorphic. Since $\varphi(M) \subseteq M$, we can define the iterates

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One of the main goals in holomorphic dynamics:

Study the asymptotic behavior of the sequence of functions $\varphi_n$. 
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\]

One of the main goals in holomorphic dynamics:

Study the asymptotic behavior of the sequence of functions \( \varphi_n \).

Key authors in the beginnings of holomorphic dynamics:
Poincaré, Julia, Fatou, Denjoy, Wolff, Carathéodory, ...

When \( M \) is the unit disk, we can cite Valiron, Baker, Pommerenke, Cowen, ... and many others authors we will mention throughout the talk.
Let $\varphi : M \to M$ be holomorphic.

**Remark**

If $g : M \to N$ is a bi-holomorphism, then

$$\psi := g \circ \varphi \circ g^{-1} : N \to N$$

is holomorphic and its iterates satisfy

$$\psi_n = (g \circ \varphi \circ g^{-1}) \circ (g \circ \varphi \circ g^{-1}) \circ \ldots \circ (g \circ \varphi \circ g^{-1})$$

$n$-times

$$= g \circ \varphi \circ \varphi \circ \ldots \circ \varphi \circ g^{-1} = g \circ \varphi_n \circ g^{-1}.$$

$\varphi_n$ has a *similar* behavior on $M$ to $\psi_n$ on $N$.

Both sequences have the same dynamics.
Introduction

In the setting of one-dimens. simply conn. complex manifolds:

By Poincaré and Koebe uniformization theorem, \( M \) is conformally equivalent to one of the following three domains:

- The Riemann sphere \( \hat{\mathbb{C}} \). Iteration of rational functions.
- The complex plane \( \mathbb{C} \). Iteration of entire functions.
  Example: \( \varphi(z) = z^2 + c \).
- The unit disk \( \mathbb{D} \).
Introduction

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  Example: $\varphi(z) = z^2 + c$.
- The unit disk $\mathbb{D}$.

Theorem (Montel)

$\Omega$ a simply connected domain in $\mathbb{C}$. $a, b \in \mathbb{C}$, $a \neq b$. Then

$$\{f : \Omega \to \mathbb{C} \setminus \{a, b\}, \text{holomorphic}\}$$

is a normal family.

If $\varphi : \mathbb{D} \to \mathbb{D}$, then the family $\{\varphi_n : n \in \mathbb{N}\}$ is normal.
In this talk, we deal with the case

$$\varphi : \mathbb{D} \to \mathbb{D}.$$ 

The behavior of $$\varphi_n(z)$$ when $$z \in \mathbb{D}$$ was studied by Denjoy and Wolff in 1926.
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Problem

Analyze the behavior of $\varphi_n(\xi)$ when $\xi \in \partial \mathbb{D}$. 
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The behavior of \(\varphi_n(z)\) when \(z \in \mathbb{D}\) was studied by Denjoy and Wolff in 1926.

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Analyze the behavior of \(\varphi_n(\xi)\) when \(\xi \in \partial \mathbb{D}\).

When \(\varphi\) is an inner function, this problem has been widely studied: Aaronson, Doering, Fernández, Hamilton, Mañé, Melián, Neuwirth, Pestana, Pommerenke, ...
In this talk, we deal with the case

\[ \varphi : \mathbb{D} \to \mathbb{D}. \]

The behavior of \( \varphi_n(z) \) when \( z \in \mathbb{D} \) was studied by Denjoy and Wolff in 1926.

**Problem**

Analyze the behavior of \( \varphi_n(\xi) \) when \( \xi \in \partial \mathbb{D} \).

When \( \varphi \) is an inner function, this problem has been widely studied: Aaronson, Doering, Fernández, Hamilton, Mañé, Melián, Neuwirth, Pestana, Pommerenke, ... If \( \varphi \) is not inner, the above problem has been treated only for a decade.
Preliminaries

Definition

\( \varphi : \mathbb{D} \rightarrow \mathbb{C} \) has angular limit \( L \in \hat{\mathbb{C}} \) at the point \( \xi \in \partial \mathbb{D} \) if for every \( \alpha < \pi/2 \)

\[
\lim_{z \rightarrow \xi, z \in S(\xi, \alpha)} \varphi(z) = L.
\]

Notation: \( \angle \lim_{z \rightarrow \xi} \varphi(z) := L. \)

Stolz angle.
Preliminaries

\( \varphi : \mathbb{D} \rightarrow \mathbb{D} \). Take \( \xi \in \partial \mathbb{D} \).
Assume that

\[
\varphi(\xi) := \angle \lim_{z \to \xi} \varphi(z) \in \partial \mathbb{D},
\]

that is, \( \xi \) is a contact point.

Then there exists the angular limit

\[
\varphi'(\xi) := \angle \lim_{z \to \xi} \frac{\varphi(z) - \varphi(\xi)}{z - \xi}.
\]

Moreover, \( \xi \varphi(\xi) \varphi'(\xi) \in (0, +\infty) \cup \{\infty\} \).

\( \varphi'(\xi) \) is called the angular derivative of \( \varphi \) at \( \xi \).

If \( \varphi(\xi) = \xi \), we say that \( \xi \) is a (boundary) fixed point.
\((\Rightarrow \varphi'(\xi) \in (0, +\infty) \cup \{\infty\})\).
The Denjoy-Wolff theorem

Theorem (Denjoy, Wolff, 1926)

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, $\varphi \neq id$ and different from an elliptic autom. (those with a fixed point in $\mathbb{D}$).
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Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be holomorphic, \( \varphi \neq \text{id} \) and different from an elliptic automorphism (those with a fixed point in \( \mathbb{D} \)).

Then there exists a point \( \tau \in \overline{\mathbb{D}} \) such that the sequence of iterates \( \varphi_n \) converges to \( \tau \) uniformly on compacta in \( \mathbb{D} \).

If \( \tau \in \partial \mathbb{D} \), then \( \tau \) is a boundary fixed point and \( \varphi'(\tau) \in (0, 1] \).
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If $\tau \in \partial \mathbb{D}$, then $\tau$ is a boundary fixed point and $\varphi'(\tau) \in (0, 1]$.

**Definition**

$\tau$ is called the **Denjoy-Wolff point** of $\varphi$. 
Boundary behavior of the iterates

Fatou: Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be hol. Then \( \varphi(\xi) := \lim_{z \to \xi} \varphi(z) \in \overline{\mathbb{D}} \) exits for almost all \( \xi \in \partial \mathbb{D} \). Moreover, the function

\[
\xi \in \partial \mathbb{D} \mapsto \varphi(\xi) \in \overline{\mathbb{D}}
\]

is measurable.

Thus, there is \( A \subseteq \partial \mathbb{D} \), with Lebesgue measure zero, s.t. \( \varphi_n(\xi) := \lim_{z \to \xi} \varphi_n(z) \) exists, for all \( \xi \in \partial \mathbb{D} \setminus A \) and for all \( n \).
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Thus, there is $A \subseteq \partial \mathbb{D}$, with Lebesgue measure zero, s.t.

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Where and in what form does the Denjoy-Wolff theorem extend to boundary points?
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Where and in what form does the Denjoy-Wolff theorem extend to boundary points?

**Theorem (Denjoy, Wolff)**

$\varphi : \mathbb{D} \rightarrow \mathbb{D}$ holomorphic, $\varphi \neq \text{id}$ and different from an elliptic automorphism. Then there exists $\tau \in \overline{\mathbb{D}}$ s.t. $\varphi_n(z) \rightarrow \tau$, for all $z \in \mathbb{D}$.

**Problem**

*Is there a set $A \subseteq \partial \mathbb{D}$, with measure zero, s.t.*

$\varphi_n(\xi) \rightarrow \tau$ for all $\xi \in \partial \mathbb{D} \setminus A$*?
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\[ \varphi : \mathbb{D} \to \mathbb{D} \text{ holomorphic,} \]
\[ \varphi \neq \text{id and different from an elliptic automorphism.} \]
Then there exists \( \tau \in \mathbb{D} \) s.t.
\[ \varphi_n(z) \to \tau, \text{ for all } z \in \mathbb{D}. \]

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Is there a set \( A \subseteq \partial \mathbb{D} \), with measure zero, s.t.
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Boundary behavior of the iterates

**Theorem (Denjoy, Wolff)**

φ : D → D holomorphic,
φ ≠ id and different from an elliptic automorphism.
Then there exists τ ∈ D s.t.
φₙ(z) → τ, for all z ∈ D.

**Problem**

Is there a set A ⊆ ∂D, with measure zero, s.t.
φₙ(ξ) → τ
for all ξ ∈ ∂D \ A?

Obviously, NO.
Boundary behavior of the iterates

**Theorem (Denjoy, Wolff)**

\( \varphi : \mathbb{D} \to \mathbb{D} \) holomorphic, \( \varphi \neq \text{id} \) and different from an elliptic automorphism. Then there exists \( \tau \in \mathbb{D} \) s.t. \( \varphi_n(z) \to \tau \), for all \( z \in \mathbb{D} \).

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\[ \varphi_n(\xi) \to \tau \]

*for all \( \xi \in \partial \mathbb{D} \setminus A \)?

Obviously, NO.

\( \varphi(z) = z^2 \). Its Denjoy-Wolff point is \( \tau = 0 \).

\[ |\varphi_n(\xi)| = |\xi^{2^n}| = 1 \not\to 0, \text{ for all } \xi \in \partial \mathbb{D}. \]

So, the “boundary Denjoy-Wolff theorem” does not hold for \( \varphi \).
Boundary behavior of the iterates

**Theorem (Denjoy, Wolff)**

$\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic,

$\varphi \neq id$ and different from an elliptic automorphism.

Then there exists $\tau \in \overline{\mathbb{D}}$ s.t.

$\varphi_n(z) \to \tau$, for all $z \in \mathbb{D}$.

**Problem**

Is there a set $A \subseteq \partial \mathbb{D}$, with measure zero, s.t.

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So, the “boundary Denjoy-Wolff theorem” does not hold for $\varphi$.

The answer depends on:
- Type of function according to the local behavior around $\tau$;
- The values of $\varphi$ on the boundary (inner functions).
Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be holomorphic.

**Classification: according to the behavior around \( \tau \).**

- **elliptic**: the ones with a fixed point inside the unit disk.
- **hyperbolic**: the ones with the Denjoy-Wolff point \( \tau \in \partial \mathbb{D} \) such that \( \varphi'(\tau) < 1 \);
- **parabolic**: the ones with the Denjoy-Wolff point \( \tau \in \partial \mathbb{D} \) such that \( \varphi'(\tau) = 1 \).

**Classification: according to the values of \( \varphi \) on \( \partial \mathbb{D} \).**

We say that \( \varphi \) is **inner** if \( \varphi(\xi) \in \partial \mathbb{D} \) for almost all \( \xi \in \partial \mathbb{D} \).
Let $\varphi : \mathbb{D} \to \mathbb{D}$ be elliptic (that is, with $\tau \in \mathbb{D}$) and non-inner.

$$J = \{ \xi \in \partial \mathbb{D} : |\varphi(\xi)| < 1 \}, \quad m(J) > 0.$$
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Theorem (Bourdon-Matache-Shapiro (2005), Poggi-Corradini (2010))

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be elliptic, $\varphi \neq id$, with Denjoy-Wolff point $\tau \in \mathbb{D}$. Then

$\varphi_n(\xi) \to \tau$ a.e. $\xi \in \partial \mathbb{D}$ if and only if $\varphi$ is not inner.
**Elliptic functions**

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- Bourdon, Matache, and Shapiro used results from Operator Theory:
**Elliptic functions**

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Assume $\tau = 0$ and $\varphi \neq \text{inner}$. Take $X = \{g \in H^2 : g(0) = 0\}$, then $M := \|C_\varphi\|_X < 1$, where $C_\varphi(g) = g \circ \varphi$. 

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\int_{\partial \mathbb{D}} \left( \sum_{n=1}^{\infty} |\varphi_n|^2 \right) dm = \sum_{n=1}^{\infty} \int_{\partial \mathbb{D}} |\varphi_n|^2 dm = \sum_{n=1}^{\infty} \| \varphi_n \|^2 \leq \sum_{n=1}^{\infty} M^{2n} < \infty.
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So \( \sum_{n=1}^{\infty} |\varphi_n(\xi)|^2 < \infty \) for almost every \( \xi \in \partial \mathbb{D} \).
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- Bourdon, Matache, and Shapiro used results from Operator Theory:
  Assume \( \tau = 0 \) and \( \varphi \neq \text{inner} \). Take \( X = \{g \in H^2 : g(0) = 0\} \), then \( M := ||C_\varphi||_X < 1 \), where \( C_\varphi(g) = g \circ \varphi \). Thus

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\int_{\partial \mathbb{D}} \left( \sum_{n=1}^{\infty} |\varphi_n|^2 \right) dm = \sum_{n=1}^{\infty} \int_{\partial \mathbb{D}} |\varphi_n|^2 dm = \sum_{n=1}^{\infty} ||\varphi_n||^2 \leq \sum_{n=1}^{\infty} M^{2n} < \infty.
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So \( \sum_{n=1}^{\infty} |\varphi_n(\xi)|^2 < \infty \) for almost every \( \xi \in \partial \mathbb{D} \).

- Poggi-Corradini used Potential Theory (Harmonic measures).
Theorem (Bourdon-Matache-Shapiro (2005), Poggi-Corradini (2010))

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be hyperbolic, with Denjoy-Wolff point $\tau \in \partial \mathbb{D}$. Then

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Main tool to tackle this result

Linear models (semiconjugations).

Roughly speaking, given $\varphi$, linear models provide a way to understand, via conjugations, the behavior of the iterates of $\varphi$ by means of iteration properties of a certain linear mapping. We will return to this notion later.

To study our problem for parabolic functions we must introduce and understand two new notions:
- hyperbolic step;
- linear models.
Notation. Denote by $\rho_D$ the hyperbolic distance in $\mathbb{D}$.

If $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\rho_D(\varphi(z), \varphi(w)) \leq \rho_D(z, w) \quad \text{for } z, w \in \mathbb{D}.$$
Hyperbolic step

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Orbit generated by $z_0 \in \mathbb{D} : z_n = \varphi_n(z_0)$. Then

$$\rho_{\mathbb{D}}(z_{n+1}, z_n) = \rho_{\mathbb{D}}(\varphi(z_n), \varphi(z_{n-1})) \leq \rho_{\mathbb{D}}(z_n, z_{n-1})$$

$$\Rightarrow \quad \rho_{\mathbb{D}}(z_{n+1}, z_n) \to q \in [0, +\infty).$$

$\varphi \neq$ elliptic autom.: the positivity of $q$ does not depend on $z_0$. 

Definition 1. $\varphi$ is of positive hyperbolic step if $q > 0$.

Definition 2. $\varphi$ is of zero hyperbolic step if $q = 0$. 

We will see that the hyperbolic step determines the behavior of the iterates on the boundary of the unit disk.
Hyperbolic step

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**Definition**

1. $\varphi$ is of positive hyperbolic step if $q > 0$.
2. $\varphi$ is of zero hyperbolic step if $q = 0$.

We will see that the hyperbolic step determines the behavior of the iterates on the boundary of the unit disk.
Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, $\varphi \neq$ elliptic autom.

Remark

1. If $\varphi$ is an automorphism, then

$$\rho_{\mathbb{D}}(z_{n+1}, z_n) = \rho_{\mathbb{D}}(\varphi(z_n), \varphi(z_{n-1})) = \rho_{\mathbb{D}}(z_n, z_{n-1}) \quad (n \in \mathbb{N}).$$

In particular, it is of positive hyperbolic step.
Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be holomorphic, \( \varphi \neq \text{elliptic autom.} \).

**Remark**

1. If \( \varphi \) is an automorphism, then

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\rho_{\mathbb{D}} (z_{n+1}, z_n) = \rho_{\mathbb{D}} (\varphi(z_n), \varphi(z_{n-1})) = \rho_{\mathbb{D}} (z_n, z_{n-1}) \quad (n \in \mathbb{N}).
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   In particular, it is of positive hyperbolic step.

2. If \( \varphi \) is elliptic, then it is of zero hyperbolic step.
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   In particular, it is of positive hyperbolic step.

2. If $\varphi$ is elliptic, then it is of zero hyperbolic step.

3. If $\varphi$ is hyperbolic, then it is of positive hyperbolic step.
Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic, $\varphi \neq$ elliptic autom.

**Remark**

1. If $\varphi$ is an automorphism, then

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   In particular, it is of positive hyperbolic step.

2. If $\varphi$ is elliptic, then it is of **zero** hyperbolic step.

3. If $\varphi$ is hyperbolic, then it is of **positive** hyperbolic step.

4. If $\varphi$ is parabolic, then it can be either of **zero** hyperbolic step or of **positive** hyperbolic step.
Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be holomorphic, \( \varphi \neq \) elliptic autom.

**Remark**

1. If \( \varphi \) is an automorphism, then
   \[
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2. If \( \varphi \) is elliptic, then it is of zero hyperbolic step.

3. If \( \varphi \) is hyperbolic, then it is of positive hyperbolic step.

4. If \( \varphi \) is parabolic, then it can be either of zero hyperbolic step or of positive hyperbolic step.

5. If \( \varphi \) is a parabolic linear fractional map, then
   \( \varphi \) is of positive hyperbolic step if and only if \( \varphi \) is an autom.
Hyperbolic step: Characterizations

The hyperbolic step is a notion that appears in quite different contexts.

**Theorem (Cowen, 1983)**

Let \( \varphi \) be a parabolic function. Then \( \varphi \) is of positive hyperbolic step if and only if there is an orbit of \( \varphi \) which is interpolating.

A sequence \((z_n) \subset \mathbb{D}\) is said to be interpolating (for \( H^\infty(\mathbb{D}) \)) if

\[
H^\infty(\mathbb{D}) \rightarrow \ell^\infty
\]

\[
f \mapsto (f(z_n))
\]

is onto.

**Carlesson:** \((z_n) \subset \mathbb{D}\) is interpolating if and only if it is uniformly separated, that is,

\[
\inf_{n \in \mathbb{N}} \prod_{j=1, j \neq n}^{\infty} \left| \frac{Z_j - Z_n}{1 - Z_j Z_n} \right| > 0.
\]
Hyperbolic step: Characterizations

Theorem (Bourdon y Shapiro, 1997)

Let $\varphi$ be parabolic with Denjoy-Wolff point $\tau$. Assume $\varphi \in C^4(\tau)$. Then

$\varphi$ is of **positive hyperbolic step** if and only if $\Re \tau \varphi''(\tau) = 0$ and $\varphi''(\tau) \neq 0$.

This result is not true if we only assume that $\varphi \in C^2(\tau)$. 
Hyperbolic step: Characterizations

Let \((\Omega, \Sigma, \mu)\) be a measure space. 

\(T : \Omega \rightarrow \Omega\) is non-singular if for any measurable subset \(A \in \Sigma\), it is satisfied that

\[ \mu(T^{-1}(A)) = 0 \text{ if and only if } \mu(A) = 0. \]

A non-singular transformation \(T\) is ergodic if

\[ T^{-1}(A) = A \text{ (mod.}\mu) \Rightarrow \mu(A) = 0 \text{ or } \mu(\Omega \setminus A) = 0. \]

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That is, any invariant subset is either the whole or the empty set (up to sets of measure zero). If \(\varphi\) is inner, we can consider \(\varphi : \partial \mathbb{D} \rightarrow \partial \mathbb{D}\).

**Theorem (Aaronson, 1981)**

*Let \(\varphi\) be a inner parabolic function. Then*

\[ \varphi \text{ is of zero hyperbolic step if and only if } \varphi \text{ is ergodic}. \]
Theorem (Hamilton, 1996)

Let \( \varphi \) be a parabolic finite Blaschke product

\[
\varphi(z) = \lambda \prod_{n=1}^{m} \frac{z_n - z}{1 - z_n z}, \quad z \in \mathbb{D}
\]

where \( \lambda \in \partial \mathbb{D} \) and \( z_1, ..., z_m \in \mathbb{D} \). Then

\( \varphi \) is of zero hyperbolic step if and only if the Julia set of \( \varphi \) is \( \partial \mathbb{D} \).
Theorem (Koenigs, 1884)

Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic with $\varphi(0) = 0$ with $\varphi'(0) = \lambda$, where $0 < |\lambda| < 1$. Then there exists a unique holomorphic function $\sigma : \mathbb{D} \rightarrow \mathbb{C}$ such that $\sigma(0) = 0$, $\sigma'(0) = 1$ and

$$\sigma \circ \varphi = \lambda \sigma.$$  \hfill (Schroeder equation)

Moreover, for all $n$, we have that $\sigma \circ \varphi^n = \lambda^n \sigma$. 
Linear fractional model: elliptic case

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Example

Take \( 0 < \lambda < 1 \).

\[
\varphi(z) := 1 - (1-z)^\lambda, \; \varphi(0) = 0, \; \varphi'(0) = \lambda,
\]

\[
\sigma(z) = \log \frac{1}{1 - z}, \; \varphi(z) = \sigma^{-1}(\lambda \sigma(z)),
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\[
\varphi_n(z) = \sigma^{-1}(\lambda^n \sigma(z)).
\]
Linear fractional model: hyperbolic and parabolic

Theorem

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic with Denjoy-Wolff point $\tau \in \partial \mathbb{D}$. Then there exists $\sigma : \mathbb{D} \to \mathbb{C}$ holomorphic such that $\sigma(0) = 0$ and

$$\sigma \circ \varphi = \sigma + 1.$$  \hspace{1cm} \text{(Abel equation)}

Moreover, for all $n$, we have $\sigma \circ \varphi_n = \sigma + n$. 

Existence: Valiron (1931), Baker and Pommerenke (1979). $\sigma$ is known as the Koenigs function associated with $\varphi$.

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Moreover, for all \( n \), we have \( \sigma \circ \varphi_n = \sigma + n \).

1. \( \varphi \) is hyperbolic if and only if \( \sigma(\mathbb{D}) \) is contained in a (horizontal) strip.

2. \( \varphi \) is parabolic with positive hyperbolic step if and only if \( \sigma(\mathbb{D}) \) is contained in a horizontal half-plane but not in a strip.

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Let $\varphi : \mathbb{D} \to \mathbb{D}$ be parabolic with positive hyperbolic step and Denjoy-Wolff point $\tau \in \partial \mathbb{D}$. Then

$$\varphi_n(\xi) \to \tau \text{ for a.e. } \xi \in \partial \mathbb{D}.$$
**Parabolic functions**

**Theorem (Bourdon-Matache-Shapiro (2005), Poggi-Corradini (2010))**

Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be parabolic with positive hyperbolic step and Denjoy-Wolff point \( \tau \in \partial \mathbb{D} \). Then

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**Theorem (Aaronson, 1978)**

Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be an inner parabolic function with Denjoy-Wolff point \( \tau \in \partial \mathbb{D} \). Then

\[ \varphi_n(\xi) \to \tau \text{ for a.e. } \xi \in \partial \mathbb{D} \text{ if and only if } \sum_{n=1}^{\infty} (1 - |\varphi_n(0)|) < +\infty. \]
### Summing up

Where and in what form does the Denjoy-Wolff theorem extend to boundary points?

**Remark:**

“Positive hyp. step” $\Rightarrow$ ‘Boundary Denjoy-Wolff theorem” holds.

<table>
<thead>
<tr>
<th>$\varphi$</th>
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</tr>
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<tbody>
<tr>
<td>Inner</td>
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**Theorem (C., Díaz-Madrigal, Pommerenke)**

Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be parabolic with zero hyperbolic step and Denjoy-Wolff point \( \tau \in \partial \mathbb{D} \). If its Koenigs function \( \sigma \) has angular limit for a.e. \( \xi \in \partial \mathbb{D} \), then

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Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be parabolic with zero hyperbolic step and Denjoy-Wolff point $\tau \in \partial \mathbb{D}$. If its Koenigs function $\sigma$ has angular limit for a.e. $\xi \in \partial \mathbb{D}$, then

$$\varphi_n(\xi) \rightarrow \tau \text{ for a.e. } \xi \in \partial \mathbb{D}.$$ 

“$\sigma$ has angular limit for a.e. $\xi \in \partial \mathbb{D}$” is satisfied if:
- $\sigma$ belongs to a Hardy space,
- $\mathbb{C} \setminus \sigma(\mathbb{D})$ has positive logarithmic capacity. In particular, if $\varphi$ is univalent.
Theorem (C., Díaz-Madrigal, Pommerenke)

Let \( \varphi \) be of zero hyperbolic step with Denjoy-Wolff point 1 and Koenigs function \( \sigma \).

\[
A = \{ \xi \in \partial \mathbb{D} \setminus \{1\} : \text{there is a curve } \Gamma \subset \mathbb{D} \text{ s.t.} \]

\[
\overline{\Gamma} = \Gamma \cup \{\xi\} \quad \text{and} \quad \inf\{\operatorname{Re} \sigma(z) : z \in \Gamma \cap \mathbb{D}\} > -\infty.
\]

Assume that there are \( (\xi^+_k) \) and \( (\xi^-_k) \) in \( A \) such that

\[
\arg \xi^+_k > 0, \quad \arg \xi^-_k < 0 \quad \text{for all } k \in \mathbb{N} \quad \text{and} \quad \lim_{k \to \infty} \xi^\pm_k = 1.
\]

If \( \xi \in A \), then

\[
\sup\{|\varphi_n(z) - 1| : z \in \Gamma \cap \mathbb{D}\} \to 0 \quad (n \to \infty).
\]
When the “boundary Denjoy-Wolff theorem” fails

Let $\varphi$ be inner that fixes the point zero and $\neq$ automorphism.

**Theorem (Pommerenke, 1981)**

Let $A, B$ be two subarcs of $\partial \mathbb{D}$. Then

$$m(B \cap \varphi^{-n}(A)) = m(\{\xi \in B : \varphi_n(\xi) \in A\}) \rightarrow m(A)m(B)$$

($\varphi$ is strongly mixing).

**Theorem (Fernández, Melián, Pestana (2007))**

Fix $\xi_0 \in \partial \mathbb{D}$ and $\alpha > 1$. Then

1. $\lim \inf_{n \to \infty} n \, d(\varphi_n(\xi), \xi_0) = 0$ for a.e. $\xi \in \partial \mathbb{D}$.
2. $\lim_{n \to \infty} n^\alpha \, d(\varphi_n(\xi), \xi_0) = +\infty$ for a.e. $\xi \in \partial \mathbb{D}$.

These results have been improved by the same authors for elliptic finite Blaschke products.
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**Open problem**

But, there is almost nothing for the parabolic case!