Well-Posedness for Degenerate Schrödinger Equations

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Outline

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Change of variable

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Concluding remarks and open problems
Joint work with

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Schrödinger equations with time-dependent Hamiltonian

We consider the Schrödinger operator

\[ S := \frac{1}{i} \partial_t - H(t) \]

with a time-dependent Hamiltonian

\[ H(t) = a(t) \Delta_x - \sum_{j=1}^{n} b_j(t, x) \partial_{x_j}, \quad a(t) \geq 0, \]

\( t \in [0, T], x \in \mathbb{R}^n \), with coefficients which are continuous in time and smooth and bounded in the space variables.
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\( t \in [0, T], x \in \mathbb{R}^n, \) with coefficients which are continuous in time and smooth and bounded in the space variables.

We are interested in the Cauchy problem

\[
\begin{cases}
Su = 0, & t > 0, \\
u(0, x) = u_0(x).
\end{cases}
\]
Well-posedness

We take Cauchy data $u_0$ in a Sobolev space $H^m$, $m \in \mathbb{R}$, or in a Gevrey-Sobolev space $H^{m,s}_\varrho$, $s > 1$, $\varrho > 0$, where

$$H^{m,s}_\varrho := e^{-\varrho (\langle D_x \rangle)^{1/s}} H^m, \quad H^{m,s} := \bigcup_{\varrho > 0} H^{m,s}_\varrho.$$
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$$H^{m,s}_{\varrho} := e^{-\varrho \langle D_x \rangle^{1/s}} H^m, \quad H^{m,s} := \bigcup_{\varrho > 0} H^{m,s}_{\varrho}.$$ 

We say that the Cauchy problem is well-posed in

$$\begin{cases} L^2 \\ H^\infty \\ H^{\infty,s} := \bigcap_m H^{m,s} \end{cases} \quad \text{when for every given } u_0 \in \begin{cases} H^m \\ H^m \\ H^{m,s}_{\varrho} \end{cases} \quad \text{there exists a unique solution } u \in \begin{cases} C([0, T]; H^m) \\ C([0, T]; H^{m-\delta}) \\ C([0, T]; H^{m',s}_{\varrho'}) \end{cases}.$$
Decay conditions

When the coefficients $b_j(t, x)$ are pure imaginary we have well-posedness without loss of derivatives by the energy method and Gronwall inequality since $H(t)$ is the sum of a self-adjoint operator and of a bounded operator in $L^2$. 
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$$\frac{1}{i} \partial_t u = \partial_x^2 u + \partial_x u$$

by Fourier transform.
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Decay conditions as $x \to \infty$ for the real parts $\Re b_j(x)$ have been proved to be necessary in the case of a time-independent Hamiltonian $H = \Delta_x + \sum_{j=1}^n b_j(x) \partial_{x_j}$, Ichinose et al.
Sufficient conditions

Still in the time-independent case, the condition

$$|\Re b_j(x)| \leq C \langle x \rangle^{-\sigma}, \sigma > 0, \langle x \rangle = \sqrt{1 + |x|^2},$$

is sufficient for the well-posedness in

$$\begin{cases}
L^2, & \sigma > 1, \\
H^\infty, & \sigma = 1, \\
H^\infty,s, & s < \frac{1}{1-\sigma}, \sigma < 1,
\end{cases}$$

Kajitani-Baba et al. These results are optimal.
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After minor changes, the same proof works for time-dependent \( H(t) \) provided that the coefficient \( a(t) \) of the Laplacian never vanishes so that \( |\Re b_j(t, x)| \leq Ca(t)\langle x\rangle^{-\sigma}. \)
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After minor changes, the same proof works for time-dependent \( H(t) \) provided that the coefficient \( a(t) \) of the Laplacian never vanishes so that \( |\Re b_j(t, x)| \leq C a(t)\langle x \rangle^{-\sigma}. \)

As far as we know, there are no well-posedness results for time-dependent Hamiltonians with \( a(t) \) that may vanish.
We consider a real coefficient \( a(t) \geq 0 \) vanishing of finite order \( \ell \) at \( t = 0 \), that is,

\[
ct^\ell \leq a(t) \leq Ct^\ell,
\]

for \( \ell \in \mathbb{R}_+ \) and positive constants \( c, C \).
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The other coefficients are complex-valued and satisfy
\[
|\Re b_j(t, x)| \leq Ct^k \langle x \rangle^{-\sigma}, \quad 0 < k \leq \ell, \quad \sigma > 0.
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The other coefficients are complex-valued and satisfy
\[ |\Re b_j(t,x)| \leq C t^k \langle x \rangle^{-\sigma}, \quad 0 < k \leq \ell, \quad \sigma > 0. \]

We have new effects for $k < \ell$ when $|\Re b_j(t,x)| \leq C a(t) \langle x \rangle^{-\sigma}$ does not hold true.
Main Result

Theorem

The Cauchy problem is well-posed in

\[
\begin{aligned}
L^2 & \text{ if } k = \ell, \sigma > 1, \\
H^\infty & \text{ if } k = \ell, \sigma = 1, \\
H^\infty_s & \text{ with } s < \frac{\ell+1}{\ell-k} \text{ if } k < \ell, \sigma \geq 1, \\
H^\infty_s & \text{ with } s < \frac{(\ell-k)\sigma+k+1}{(\ell-k)\sigma+(k+1)(1-\sigma)} \text{ if } k \leq \ell, \sigma < 1.
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\end{align*}
\]

For any $\ell \geq 0$ and $k = \ell$ we have the same optimal spaces of well-posedness as in the time-independent case.
Transforming $iH(t)$ into a bounded from above operator

We get the well-posedness of the Cauchy problem after performing a change of variable $v(t, x) = e^\Lambda(t, x, D_x)u(t, x)$, where $e^\Lambda(t, x, D_x)$, $D = \frac{1}{i} \partial$, is an invertible pseudo-differential operator with symbol $e^{\Lambda(t,x,\xi)}$, $\Lambda(t, x, \xi)$ real-valued of order $q$, $0 \leq q < 1$. 
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$$\|\nu(t)\|_{L^2} \leq C\|\nu(0)\|_{L^2}$$

for any solution of the transformed equation

$$S_\Lambda \nu = 0, \quad S_\Lambda := e^\Lambda S(e^\Lambda)^{-1}.$$
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for any solution of the transformed equation

$$S_\Lambda v = 0, \quad S_\Lambda := e^\Lambda S(e^\Lambda)^{-1}.$$

The energy estimate (without any loss of regularity) follows by Gronwall’s lemma if we find $\Lambda$ such that

$$iS_\Lambda = \partial_t - ia(t)\Delta_x - A(t, x, D_x), \quad 2\Re(Av, v) \leq C\|v\|_{L^2}^2.$$
The crucial inequality for the symbol $\Lambda$

We seek for a function $\Lambda$ that solves

$$\partial_t \Lambda(t, x, \xi) + 2a(t) \sum_{j=1}^{n} \xi_j \partial_{x_j} \Lambda(t, x, \xi) + \Re \sum_{j=1}^{n} b_j(t, x) \xi_j \leq 0,$$

for all $|\xi| \geq h$, and such that $\partial_t \Lambda(t, x, \xi)$ has the order 1 and $a(t)\partial_{x_j} \Lambda$ has the order zero.
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for all $|\xi| \geq h$, and such that $\partial_t \Lambda(t, x, \xi)$ has the order 1 and $a(t) \partial_{x_j} \Lambda$ has the order zero. This means that we make $A(t)$ an operator of order 1 with negative principal symbol. In view of the sharp Gårding inequality, this leads to a bounded from above operator.
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This means that we make $A(t)$ an operator of order 1 with negative principal symbol. In view of the sharp Gårding inequality, this leads to a bounded from above operator. As it is well-known, then the energy estimate gives the well-posedness in $L^2$ of the Cauchy problem for the operator $S_\Lambda$. 
The transformation carries the loss

If $X$ is a suitable Banach or Frechet space of functions on $\mathbb{R}^n_X$ such that the operators

$$e^{\Lambda(t)} : X \to H^m, \quad (e^{\Lambda(t)})^{-1} : H^m \to X,$$

are continuous, then we have (at least locally in time) a unique solution $u \in C([0, T]; X)$ of the original Cauchy problem for any given initial data $u_0 \in X$. The order of $e^{\Lambda}$ corresponds to the loss of derivatives and determines the space $X$. 
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We obtain spaces of well-posedness from the following estimates:

$$|\Lambda(t, x, \xi)| \leq \begin{cases} 
C\langle\xi\rangle^{\frac{\ell-k}{\ell+1}} & \text{if } \sigma > 1, \\
C\langle\xi\rangle^{\frac{\ell-k}{\ell+1}} \log(1 + \langle\xi\rangle) & \text{if } \sigma = 1, \\
C\langle\xi\rangle^{\frac{(\ell-k)\sigma+(k+1)(1-\sigma)}{(\ell-k)\sigma+k+1}} & \text{if } \sigma < 1.
\end{cases}$$
Degeneracy leads to solvability in Gevrey spaces

In particular, when $k = \ell$ the operator $e^{\Lambda}$ is:

- of order zero for $\sigma > 1$, $X$ is the Banach space $H^m$;
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- of order zero for \( \sigma > 1 \), \( X \) is the Banach space \( H^m \);
- of a finite positive order \( \delta \) for \( \sigma = 1 \), \( X \) is the Frechet space \( H^\infty \) (with a loss of \( \delta \) derivatives);
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- of a finite positive order $\delta$ for $\sigma = 1$, $X$ is the Frechet space $H^\infty$ (with a loss of $\delta$ derivatives);
- of infinite order described by the symbol $e^{\rho \langle \xi \rangle^{1-\sigma}}$ for $\sigma < 1$, $X$ is the Frechet space $H^{m,s}$, $s = 1/(1 - \sigma)$. 

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In this case, we have the same spaces of well-posedness as in the time-independent case.
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In this case, we have the same spaces of well-posedness as in the time-independent case.

- For $k < \ell$ the operator $e^\Lambda$ is of infinite order described by $e^{\rho\langle \xi \rangle^q}$, $0 < q < 1$, $q = q(\ell, k, \sigma)$, even with a fast decay $\sigma > 1$. A strong degeneracy leads to well-posedness only in Gevrey classes of index $s \leq 1/q$. 

Solving modulo a prescribed order

Let us devote to the inequality which is to be satisfied by $\Lambda$. Let $w(\xi)$ be a weight function corresponding to a possible order of solutions. It is sufficient to find $\lambda(t, x, \xi)$ of the same order as that of $w(\xi)$ such that

$$\partial_t \lambda(t, x, \xi) + 2a(t) \sum_{j=1}^n \xi_j \partial_{x_j} \lambda(t, x, \xi) + \Re \sum_{j=1}^n b_j(t, x) \xi_j \leq Kw(\xi).$$
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\]

In fact, if we define \( \Lambda(t, x, \xi) \) by

\[
\Lambda(t, x, \xi) = \varrho(t)w(\xi) + \lambda(t, x, \xi),
\]

then we have a solution of still of the order of \( w(\xi) \) taking \( \varrho(t) \) such that \( \varrho'(t) \leq -K \).
Absorbing lower order terms

It is natural to absorb an error of the order of $w(\xi)$ because terms of such an order appear under the principal part in the asymptotic expansion of the operator $A(t)$ in any case. If $w(\xi)$ is not of order zero, then we also need to control them in the application of the Gårding inequality.
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The symbol of this part of the order of $w(\xi)$ will be bounded by $N|\varrho(t)| + N$, $N \geq K$, so we will choose $\varrho(t)$ as a solution of

$$\varrho'(t) + N(\varrho(t) + 1) = 0, \quad \varrho(t) > 0.$$
Splitting the phase-space

The study of the inequality for $\lambda(t, x, \xi)$ is crucial in the zone

$$\{(x, \xi) \in \mathbb{R}^{2n}_{x, \xi} : \langle x \rangle \leq \langle x \xi \rangle \text{ with } \langle x \xi \rangle = \langle \xi \rangle^{(1-q)/\sigma}\}$$

of the phase-space $\mathbb{R}^{2n}_{x, \xi}$ since we have in the other part

$$\sum_{j=1}^{n} |\Re b_j(t, x)\xi_j| \leq Ct^k \langle \xi \rangle^q, \quad \langle x \rangle \geq \langle x \xi \rangle.$$

So, we can use here the above absorbing argument.
The solution for mild degeneracy

For $k = \ell$ we have

$$a(t)M_0|\xi|\langle x\rangle^{-\sigma} \geq \sum_{j=1}^{n} |\Re b_j(t, x)\xi_j|.$$
The solution for mild degeneracy

For $k = \ell$ we have

$$a(t) M_0 |\xi| \langle x \rangle^{-\sigma} \geq \sum_{j=1}^{n} |\Re b_j(t, x) \xi_j|.$$ 

In this case we can take a time-independent solution $\lambda_0(x, \xi)$

$$\sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda_0(x, \xi) + M |\xi| \langle x \rangle^{-\sigma} \chi \left( \frac{\langle x \rangle}{\langle x \xi \rangle} \right) \leq 0,$$

where $\chi(y)$ is a cut-off function.
The solution for mild degeneracy

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In this case we can take a time-independent solution $\lambda_0(x, \xi)$

$$\sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda_0(x, \xi) + M|\xi|\langle x \rangle^{-\sigma} \chi(\langle x \rangle / \langle x\xi \rangle) \leq 0,$$

where $\chi(y)$ is a cut-off function.

The equation $\sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda(x, \xi) + |\xi|g(x, \xi) = 0$ is solved by

$$\lambda(x, \xi) = -\int_{0}^{x \cdot \omega} g(x - \tau \omega, \xi) d\tau, \quad \omega = \xi/|\xi|.$$
The order of the time-independent solution

We have

\[ |\lambda_0(x, \xi)| \leq \begin{cases} 
  C_0 \langle \xi \rangle^{(1-q)(1-\sigma)/\sigma}, & \sigma < 1, \\
  C_0 \log(1 + \langle \xi \rangle), & \sigma = 1, \\
  C_0, & \sigma > 1. 
\end{cases} \]
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C_0, & \sigma > 1.
\end{cases}
\]

For $\ell = k$ the optimal choice of the order $q$, together with the related Gevrey index $s < 1/q$ for $q > 0$, follows from

\[
\begin{cases} 
(1 - q)(1 - \sigma)/\sigma = q, & \sigma \in (0, 1), \\
q = 0, & \sigma \geq 1.
\end{cases}
\]

The first line gives $q = 1 - \sigma$ for $\sigma < 1$.  

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Degenerate Schrödinger
Strong degeneracy - Splitting the extended phase-space

For $k < \ell$ we split the extended phase-space $(t, x, \xi)$ into two zones. Defining $t_\xi = \langle \xi \rangle^{-(1-q)/(k+1)}$ we introduce the
• pseudo-differential zone: $t \leq t_\xi$; evolution zone: $t \geq t_\xi$. 
Strong degeneracy - Splitting the extended phase-space

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- pseudo-differential zone: $t \leq t_\xi$; evolution zone: $t \geq t_\xi$.

We put in the construction of a solution $\lambda(t, x, \xi)$ a first term

$$\lambda_\psi(t, \xi) = -M\langle \xi \rangle \int_0^t \tau^k \chi(\tau/t_\xi) d\tau$$

which is localized to the pseudo-differential zone. The symbol $\lambda_\psi(t, \xi)$ is of order $q$ by the definition of $t_\xi$. 
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$$

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Taking a sufficiently large $M$ it follows

$$
\partial_t \Lambda_\psi(t, \xi) - \chi(t/t_\xi)\Re \sum_{j=1}^n b_j(t, x) \xi_j \leq 0
$$

since

$$
\sum_{j=1}^n |\Re b_j(t, x) \xi_j| \leq M_0 t^k \langle x \rangle^{-\sigma} |\xi| \leq M_0 t^k |\xi|.
$$
The solution in the evolution zone

In the evolution zone we define

$$\lambda_e(t, x, \xi) = \lambda_{e,0}(t, x, \xi) + \lambda_{e,1}(t, \xi)$$

with

$$\lambda_{e,0}(t, x, \xi) = (1 - \chi(t/t\xi)) t^{k-\ell} \lambda_0(x, \xi),$$

$$\lambda_{e,1}(t, \xi) = -C_1 M w(\xi) \int_0^t \tau^{k-\ell-1} (1 - \chi(2\tau/t\xi)) d\tau$$

where $\lambda_0(x, \xi)$ is the time independent solution for $k = \ell$ and the weight function

$$w(\xi) = \begin{cases} 
\langle\xi\rangle^{(1-q)(1-\sigma)}/\sigma, & \sigma < 1, \\
\log\langle\xi\rangle, & \sigma = 1, \\
1, & \sigma > 1,
\end{cases}$$

gives its order before fixing $q$. 
Fixing the order

We have a solution in the evolution zone as soon as 
\( \partial_t \lambda_e(t, x, \xi) \leq 0 \). We have this fixing a large constant \( C_1 \) in the correction term \( \lambda_{e,1}(t, \xi) \).
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Using the definitions of \( t_\xi \) and the order of the time-independent term \( \lambda_0(x, \xi) \), the symbol \( \lambda_e(t, x, \xi) \) can be estimated by

\[
\begin{cases}
\langle \xi \rangle (1-q)((\ell-k)/(k+1)+(1-\sigma)/\sigma), & \sigma < 1, \\
\langle \xi \rangle (1-q)(\ell-k)/(k+1) \log \langle \xi \rangle, & \sigma = 1, \\
\langle \xi \rangle (1-q)(\ell-k)/(k+1), & \sigma > 1.
\end{cases}
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Fixing the order

We have a solution in the evolution zone as soon as $\partial_t \lambda_e(t, x, \xi) \leq 0$. We have this fixing a large constant $C_1$ in the correction term $\lambda_{e,1}(t, \xi)$.

Using the definitions of $t_{\xi}$ and the order of the time-independent term $\lambda_0(x, \xi)$, the symbol $\lambda_e(t, x, \xi)$ can be estimated by

$$
\begin{cases}
\langle \xi \rangle (1-q)((\ell-k)/(k+1)+(1-\sigma)/\sigma), & \sigma < 1, \\
\langle \xi \rangle (1-q)(\ell-k)/(k+1) \log \langle \xi \rangle, & \sigma = 1, \\
\langle \xi \rangle (1-q)(\ell-k)/(k+1), & \sigma > 1.
\end{cases}
$$

In order to have also $\lambda_e$ of order $q$ (or $q \log$ for $\sigma = 1$) we choose

$$
q = \begin{cases}
\frac{(\ell-k)\sigma+(k+1)(1-\sigma)}{(\ell-k)\sigma+k+1}, & \sigma < 1, \\
\frac{\ell-k}{\ell+1}, & \sigma \geq 1.
\end{cases}
$$
The transformed Cauchy problem

For operators of infinite order of Gevrey type, we use the calculus of Kajitani and Nishitani. Localizing the support of \( \lambda(t, x, \xi) \) for \( |\xi| \geq h \) with a sufficiently large \( h \), we can make the change of variable \( v = e^{\Lambda} u \) invertible with \( (e^{\Lambda})^{-1} \) given by a Neumann series of operators.
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$$iS_\Lambda = \partial_t - ia(t)\Delta_x - A(t, x, D_x), \quad 2\Re(Av, v) \leq C\|v\|_{L^2}^2.$$  

This gives the energy estimate without loss of derivatives hence the well-posedness in $L^2$ of the Cauchy problem for $S_\Lambda$. 
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This gives the energy estimate without loss of derivatives hence the well-posedness in $L^2$ of the Cauchy problem for $S_\Lambda$. Taking the order of $e^\Lambda$ into account (the transformation carries the loss) we have the results of well-posedness for the operator $S$. 

Degenerate Schrödinger
General degeneracy

Let us consider a general coefficient $a(t)$ increasing and such that $a(t) = 0$ (also of infinite order) and let us assume

$$|\Re b_j(t, x)| \leq Ca(t)\mu(t)\langle x\rangle^{-\sigma}, \quad \sigma > 0,$$

with $\mu(t)$ decreasing and such that $\lim_{t \to +0} \mu(t) = \infty$. 
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The separating line $t = t_\xi$ in the extended phase-space is now defined by

$$B(t_\xi) = \langle \xi \rangle^{q-1}, \quad \text{where } B(t_\xi) := \int_0^{t_\xi} a(\tau) \mu(\tau) d\tau, \quad q < 1.$$
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In the definition of $\lambda_e(t, x, \xi)$ the factor $t^{k-\ell}$ is replaced by $\mu(t)$. Computing the lowest possible order of $\lambda = \lambda_\psi + \lambda_e$ we can get results of well-posedness.
A model with degeneracy of infinite order

Let us take

\[ a(t) = \frac{\alpha}{t^{\alpha+1}} \exp \left( - \frac{1 + c_0}{t^\alpha} \right), \]

\[ \mu(t) = \exp \left( \frac{c_0}{t^\alpha} \right), \]

and let us assume, consequently,

\[ |\Re b_j(t, x)| \leq C \frac{\alpha}{t^{\alpha+1}} \exp \left( - \frac{1}{t^\alpha} \right) \langle x \rangle^{-\sigma} \text{ with } \sigma \in (0, 1). \]
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The critical order follows from the balance

\[ q = \frac{(1-q)(1-\sigma)}{\sigma} + (1-q)c_0. \]

Then the Cauchy problem is well-posed in the Gevrey spaces \( H^{\infty,s} \) with \( s < \frac{1+c_0\sigma}{1-\sigma+c_0\sigma}. \)
Vibrating plates

Let us consider the vibrating plate equation $Pu = 0$

$$Pu := u_{tt} + a^2(t)\Delta^2_x u + \sum_{|\alpha|\leq 3} b_\alpha(t, x)\partial_x^\alpha u$$

with $a(t) \geq 0$ vanishing at $t = 0$ of finite order $\ell$ and with real-valued $b_\alpha(t, x)$ with $|\alpha| = 3$ satisfying

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The operator $P$ can be formally factorized in the product of two (pseudo-differential) Schrödinger operators

$$P = S_+ S_-, \quad S_\pm = \partial_t \pm ia(t)\Delta_x \pm b(t, x, \partial_x)$$

modulo terms of order 2.
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P = S_+ S_-, \quad S_\pm = \partial_t \pm ia(t) \Delta_x \pm b(t, x, \partial_x)
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modulo terms of order 2.
Performing a complete diagonalization in the evolution zone one should obtain for \( P \) the same results as for \( S \) with \( k = j - \ell \).
Necessity

An interesting problem is the optimality of the results, a subject widely studied for non-degenerate models. One can not find “better” spaces of well-posedness in the case $\ell = k$ in view of the necessary decay conditions as $x \to \infty$ obtained for $\ell = k = 0$. 
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THANKS FOR YOUR ATTENTION!!!